On a special second-order differential subordination using $q-$analogue of Noor integral operator

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Abstract

In the present paper we establish a new and special differential subordination regardind the q-analoque of Noor integral operator. A new and interesting class of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of differential subordination we will derive different properties and characteristics of the new class.

2000 Mathematics Subject Classification. **30C45**. 30C50, 47B38.
Keywords. analytic functions, univalent function, differential subordination, convex, $q-$analogue of Noor integral operator, dominant.

1 Introduction

Let the notation $\mathcal{H}(\ \mathbb{U})$ indicate the family of holomorphic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} = \mathbb{U}\backslash\{0\}$. For $n \in N$ and $a \in \mathbb{C}$, we demonstrate by

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H}(\ \mathbb{U}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots , z \in \mathbb{U}\right\},$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0,1]$, $\mathcal{H} \equiv \mathcal{H}[1,1]$, $\mathrm{A}_n \subset \mathcal{H}_0$

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(\ \mathbb{U}) : f(z) = z + a_n z^n + a_{n+1}z^{n+1} + \ldots , z \in \mathbb{U}\right\} \tag{1.1}$$

with A_1 =A. Let S denote the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{U}$. If a function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a convex domain and f is univalent , then f is called a convex function. Let

$$\mathcal{C} = \left\{f \in \mathcal{A} : \Re\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > 0, z \in \mathbb{U}\right\}$$

denote the class of all convex functions defined in $\mathbb{U}$ and normalized by $f(0) = 0, f^{'}(0) = 1$.

Let f and g are in the class $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to g, if there exists a Schwartz function w analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \qquad (z \in \mathbb{U}).$$

such that

$$f(z) = g(w(z)).$$

Tbilisi Mathematical Journal Special Issue (8 - 2021), pp. 109–123.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 13 May 2020.
Accepted for publication: 23 December 2020.

In such a case we write

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g.$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence ([7] and [11])

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1.1. For $q \in (0,1)$, the q-derivative of function $f \in$A is defined by (see[19])

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \tag{1.2}$$

and

$$\partial_q f(0) = f'(0).$$

Thus we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k,q]\, a_k z^{k-1} \tag{1.3}$$

where $[k,q]$ is given by

$$[k,q] = \frac{1-q^k}{1-q}, \qquad [0,q] = 0 \tag{1.4}$$

and the q-fractional is defined by

$$[k,q]! = \begin{cases} \prod_{n=1}^{k} [n,q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{1.5}$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p} \geq 0$ is given by

$$[\mathfrak{p},q]_k = \begin{cases} \prod_{n=1}^{k} [\mathfrak{p}+n-1,q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{1.6}$$

As $q \to 1$, then we get $[k,q] \to k$. Thus, by choosing the function $g(z) = z^k$, while $q \to 1$, then we obtain

$$\partial_q g(z) = \partial_q z^k = [k,q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Recently, function $F^{-1}_{q,\mu+1}(z)$ is defined by Arif et al. [1] as the following relation

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z\partial_q f(z), \quad (\mu > -1) \tag{1.7}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k-1,q]!} z^k, \quad z \in \mathcal{D}. \tag{1.8}$$

To the series defined in (1.8) is convergent absolutely in $z \in \mathfrak{D}$, by using the definition of q-derivative through convolution, we now explaine the integral operator $\zeta_q^{\mu} : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^{\mu} f(z) = F_{q,\mu+1}^{-1}(z) * f(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{D}) \tag{1.9}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{1.10}$$

From (1.9), we can easily obtain the identity

$$[\mu+1,q]\zeta_q^{\mu} f(z) = [\mu,q]\zeta_q^{\mu+1} f(z) + q^{\mu} z \partial_q(\zeta_q^{\mu+1} f(z)). \tag{1.11}$$

We can state that

$$\zeta_q^0 f(z) = z\partial_q f(z), \;\; \zeta_q' f(z) = f(z) \tag{1.12}$$

also

$$\lim_{q\to 1^-} \zeta_q^{\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{1.13}$$

This means that, by taking $q \to 1$, the operator defined in (1.9) reduces to the famous Noor integral operator given in ([14, 15]). Also for more details on the q-analogue of differential and integral operators, see the work of Aldweby and Darus (see[5]).

The method of differential subordinations (also known the admissible functions method) was first introduced by Miller and Mocanu in 1978 [12] and the theory started to develop in 1981[13]. All the details captured in a book by Miller and Mocanu in 2000 [11]. Recent years, many authors investigated properties of differantial subordinations ([17] [**?**], [9], [2], [3] and others).

Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \longrightarrow \mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and and satisfies the second-order differential subordination

$$\Psi\left(p(z), zp^{'}(z), zp^{''}(z); z\right) \prec h(z) \tag{1.14}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominanat of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all p satisfying(1.14) A dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (1.14), is said to be the best dominant of (1.14).

Definition 1.2. Let $\Re_{\mu,q}(\xi)$ be the class of functions $f \in$A satisfying

$$\Re\left\{\left(\zeta_q^{\mu} f(z)\right)^{'}\right\} > \xi \tag{1.15}$$

where $z \in \mathbb{U}$, $0 \leq \xi < 1$ and ζ_q^{μ} is the q-analagoue of Noor integral operator.

In order to prove our main results, we will need the following lemmas

Lemma 1.3. [10] Let h be convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in \mathcal{H}[a, n]$ and

$$p(z)+\frac{1}{\gamma}zp^{'}(z)\prec h(z), \tag{1.16}$$

then

$$p(z)\prec q(z)\prec h(z), \tag{1.17}$$

where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z t^{\gamma/n-1}h(t)dt \qquad (z\in\mathbb{U}). \tag{1.18}$$

The function q is convex and is the best dominant of the subordination (1.16).

Lemma 1.4. [16] Let $\Re\{m\}>0, n\in\mathbb{N}$ and let

$$w=\frac{n^2+|m|^2-\left|n^2-m^2\right|}{4n\Re\{m\}}. \tag{1.19}$$

Let h be an analytic function in $\mathbb{U}$ with $h(0)=1$ and assume that

$$\Re\left\{1+\frac{zh^{''}(z)}{h^{'}(z)}\right\}>-w.$$

If

$$p(z)=1+p_nz^n+p_{n+1}z^{n+1}+\ldots$$

is analytic in $\mathbb{U}$ and

$$p(z)+\frac{1}{m}zp^{'}(z)\prec h(z), \tag{1.20}$$

then

$$p(z)\prec q(z) \tag{1.21}$$

where q is a solution of the differential equation

$$q(z)+\frac{n}{m}zq^{'}(z)=h(z),\quad q(0)=1,$$

given by

$$q(z)=\frac{m}{nz^{m/n}}\int_0^z t^{m/n-1}h(t)dt \qquad (z\in\mathbb{U}). \tag{1.22}$$

Also q is the best dominant of the differential subordination (1.20).

Lemma 1.5. [18] Let r be a convex function in $\mathbb{U}$ and let

$$h(z)=r(z)+n\xi zr^{'}(z)\quad,\quad (z\in\mathbb{U})$$

where $\xi>0$ and $n\in\mathbb{N}$. If

$$p(z)=r(0)+p_nz^n+p_{n+1}z^{n+1}+...,\quad (z\in\mathbb{U})$$

is holomorphic in $\mathbb{U}$ and

$$p(z)+\xi zp^{'}(z)\prec h(z),\quad (z\in\mathbb{U}),$$

then

$$p(z)\prec r(z),$$

and this result is sharp.

In this work, by utilizing the subordination results of [10] and [16] we will prove our main results.

2 Main results

Theorem 2.1. The set $\Re_{\mu,q}(\xi)$ is convex.

Proof. Let

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k\qquad (z\in\mathbb{U};\ j=1,....m) \tag{2.1}$$

be in the class $\Re_{\mu,q}(\xi)$.Then, by the Definition 1.2, we have

$$\Re\left\{\left(\zeta_j^{\mu}f(z)\right)^{'}\right\}=\Re\left\{1+\sum_{k=2}^{\infty}k\phi_{k-1}a_{k,j}z^{k-1}\right\}>\xi \tag{2.2}$$

For any positive numbers λ_1 ,$\lambda_2,...,\lambda_m$ such that

$$\sum_{j=1}^{m}\lambda_j=1, \tag{2.3}$$

We have to show that the function

$$h(z)=\sum_{j=1}^{m}\lambda_jf_j(z) \tag{2.4}$$

is in the class $\Re_{\mu,q}(\xi)$; that is,

$$\Re\left\{\left(\zeta_q^{\mu}h(z)\right)^{'}\right\}>\xi. \tag{2.5}$$

Thus, we get

$$\zeta_q^{\mu}h(z)=z+\sum_{k=2}^{\infty}\phi_{k-1}\left(\sum_{j=1}^{m}\lambda_ja_{k,j}\right)z^k. \tag{2.6}$$

Differentiating eq. (2.6) with respect to z , we obtain

$$\left(\zeta_q^{\mu}h(z)\right)^{'}=1+\sum_{k=2}^{\infty}k\phi_{k-1}\left(\sum_{j=1}^{m}\lambda_ja_{k,j}\right)z^{k-1} \tag{2.7}$$

and we have

$$\Re\left\{\left(\zeta_q^{\mu}h(z)\right)^{'}\right\}\ =\ 1+\sum_{j=1}^{m}\lambda_j\Re\left\{\sum_{k=2}^{\infty}k\phi_{k-1}a_{k,j}z^{k-1}\right\}$$

$$\begin{aligned} &> 1+\sum_{j=1}^{m}\lambda_j(\xi-1), \quad (by\,(2.5)) \\ &= \xi. \end{aligned} \tag{2.8}$$

Thus, the inequality (2.2) holds and we get desired result. ∎

Theorem 2.2. Let q be convex function in $\mathbb{U}$ with $q(0)=1$ and let

$$h(z)=q(z)+\frac{1}{\gamma+1}zq^{'}(z) \quad (z\in\mathbb{U}), \tag{2.9}$$

where γ is a complex number with $\Re\{\gamma\}>-1$. If $f\in\Re_{\mu,\theta}(\xi)$ and $\mathcal{G}=\Upsilon_\gamma f$, where

$$\mathcal{G}(z)=\Upsilon_\gamma f(z)=\frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \tag{2.10}$$

then,

$$\left(\zeta_q^\mu f(z)\right)^{'}\prec h(z) \tag{2.11}$$

implies

$$\left(\zeta_q^\mu \mathcal{G}(z)\right)^{'}\prec q(z), \tag{2.12}$$

and this result is sharp.

Proof. From the equality (2.10), we can write

$$z^\gamma\mathcal{G}(z)=(\gamma+1)\int_0^z t^{\gamma-1}f(t)dt, \tag{2.13}$$

By differentiating (2.13) with respect to z we obtain

$$(\gamma)\,\mathcal{G}(z)+z\mathcal{G}^{'}(z)=(\gamma+1)\,f(z), \tag{2.14}$$

and by applying the operator ζ_q^μ to the last equation, we have

$$(\gamma)\,\zeta_q^\mu\mathcal{G}(z)+z\left(\zeta_q^\mu\mathcal{G}(z)\right)^{'}=(\gamma+1)\,\zeta_q^\mu f(z). \tag{2.15}$$

If we differentiate (2.15) with respect to z, we can obtain

$$\left(\zeta_q^\mu\mathcal{G}(z)\right)^{'}+\frac{1}{\gamma+1}z\left(\zeta_q^\mu\mathcal{G}(z)\right)^{''}=\left(\zeta_q^\mu f(z)\right)^{'}. \tag{2.16}$$

Utilizing the differential subordination given by (2.11) in the equality (2.16), we get

$$\left(\zeta_q^\mu\mathcal{G}(z)\right)^{'}+\frac{1}{\gamma+1}z\left(\zeta_q^\mu\mathcal{G}(z)\right)^{''}\prec h(z). \tag{2.17}$$

Now, we define

$$p(z) = \left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} . \tag{2.18}$$

Then by a simple computation we have

$$\begin{aligned} p(z) &= \left[z + \sum_{k=2}^{\infty} \frac{\gamma+1}{\gamma+k}\phi_{k-1}a_k z^k\right]^{'} \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in H[1,1]) . \end{aligned} \tag{2.19}$$

Using (2.18) in the subordination (2.17), we obtain

$$\begin{aligned} p(z) + \frac{1}{\gamma+1} z p^{'}(z) &\prec h(z) \\ &= q(z) + \frac{1}{\gamma+1} z q^{'}(z) \qquad (z \in \mathbb{U}). \end{aligned} \tag{2.20}$$

If we use Lemma 1.4, then we write

$$p(z) \prec q(z). \tag{2.21}$$

So we obtain the desired result and q is the best dominant. ∎

Example 2.3. If we choose in Theorem 2.1

$$\gamma = i + 1, \quad q(z) = \frac{1+z}{1-z} \tag{2.22}$$

thus we get

$$h(z) = \frac{(i+2) - ((i+2)-1)z}{(i+2)(1-z)^2}.$$

If $f \in \Re_{\mu,\theta}(\xi)$ and $\mathcal{G}$ is given by

$$\mathcal{G}(z) = \Upsilon_i f(z) = \frac{i+2}{z^{i+1}} \int_0^z t^i f(t) dt, \tag{2.23}$$

then by theorem 2.2,we obtain

$$\begin{aligned} \left(\zeta_q^{\mu} f(z)\right)^{'} \quad &\prec \quad h(z) = \frac{(i+2) - ((i+2)-1)z}{(i+2)(1-z)^2} \\ &\Longrightarrow \quad \left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} \prec \frac{1+z}{1-z}. \end{aligned} \tag{2.24}$$

Theorem 2.4. Let $\Re\{\gamma\} > -1$ and let

$$w = \frac{1 + |\gamma+1|^2 - \left|\gamma^2 + 2\gamma\right|}{4\Re\{\gamma+1\}}.$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\Re\left\{1+\frac{zh^{''}(z)}{h^{'}(z)}\right\}>-w.$$

If $f\in\Re_{\mu,\theta}(\xi)$ and $\mathcal{G}=\Upsilon_{\gamma}^{\theta}\zeta_{q}^{\mu}$, where $\mathcal{G}$ is defined by (2.10), then

$$\left(\zeta_q^{\mu}f(z)\right)^{'}\prec h(z) \tag{2.25}$$

implies

$$\left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'}\prec q(z)$$

where q is the solution of the differential equation

$$h(z)=q(z)+\frac{1}{\gamma+1}zq^{'}(z),\qquad q(0)=1,$$

given by

$$q(z)=\frac{\gamma+1}{z^{\gamma+1}}\int_0^z t^{\gamma}f(t)dt. \tag{2.26}$$

Moreover q is the best dominant of the subordination (2.25) .

Proof. If we take $n=1$ and $m=\gamma+1$ in lemma 1.4, then the proof is hold by means of the Theorem2.2. ∎

Theorem 2.5. Let

$$h(z)=\frac{1+(2\xi-1)z}{1+z},\quad 0\leq\xi<1 \tag{2.27}$$

be convex in $\mathbb{U}$,with $h(0)=1$ and $0\leq\xi<1$. If $f\in\mathcal{A}$ and verifies the differential subordination

$$\left(\zeta_q^{\mu}f(z)\right)^{'}\prec h(z), \tag{2.28}$$

then

$$\begin{aligned}\left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} &\prec q(z)\\ &= (2\xi-1)+\frac{2(1-\xi)(\gamma+1)\phi(\gamma)}{z^{\gamma+1}}.\end{aligned} \tag{2.29}$$

Where ϕ is given by

$$\phi(\gamma)=\int_0^z\frac{t^{\gamma}}{t+1}dt \tag{2.30}$$

and $\mathcal{G}$ given by equation (2.10) .The function q is convex and is the best dominant.

Proof. If

$$h(z)=\frac{1+(2\xi-1)z}{1+z},\quad 0\leq\xi<1$$

then h is convex and by means of Theorem(2.4) ,we have

$$\left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} \prec q(z).$$

By using lemma 1.3 we get

$$\begin{aligned} q(z) &= \frac{\gamma+1}{z^{\gamma+1}}\int_0^z t^{\gamma}h(t)dt \\ &= \frac{\gamma+1}{z^{\gamma+1}}\int_0^z t^{\gamma}\left[\frac{1+(2\xi-1)t}{1+t}\right]dt \\ &= (2\xi-1)+\frac{2(1-\xi)(\gamma+1)}{z^{\gamma+1}}\phi(\gamma). \end{aligned}$$

Where ϕ is given by (2.30) , so we get

$$\begin{aligned} \left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} &\prec q(z) \\ &= (2\xi-1)+\frac{2(1-\xi)(\gamma+1)\phi(\gamma)}{z^{\gamma+1}}. \end{aligned}$$

The function q is convex and is the best dominant. ∎

Theorem 2.6. If $0 \le \xi < 1, \mu > -1$, $\Re\{\gamma\} > -1$ and $\mathcal{F} = \Upsilon_{\gamma} f$ is defined by (2.10), then we have

$$\Upsilon_{\gamma}\left(\Re_{\mu,q}(\xi)\right) \subset \Re_{\mu,q}(\rho), \tag{2.31}$$

where

$$\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma,\xi) = (2\xi-1)+2(1-\xi)(\gamma+1)\phi(\gamma) \tag{2.32}$$

and ϕ is given by (2.30).

Proof. Let h is given by the equation (2.27) , $f \in \Re_{\mu,q}(\xi)$ and $\mathcal{F} = \Upsilon_{\gamma} f$ is defined by (2.10) . Then h is convex and by Theorem(2.4) , we deduce

$$\begin{aligned} \left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'} &\prec q(z) \\ &= (2\xi-1)+\frac{2(1-\xi)(\gamma+1)\phi(\gamma)}{z^{\gamma+1}}. \end{aligned} \tag{2.33}$$

where ϕ is given by (2.30) . Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis and $\Re\{\gamma\} > -1$, we have

$$\begin{aligned} \Re\left\{\left(\zeta_q^{\mu}\mathcal{G}(z)\right)^{'}\right\} &\ge \min_{|z|=1}\Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma,\xi) \\ &= (2\xi-1)+2(1-\xi)(\gamma+1)(1-\xi)\phi(\gamma). \end{aligned}$$

From the inequality (2.33), we get

$$\Upsilon_{\gamma}\left(\Re_{\mu,q}(\xi)\right) \subset \Re_{\mu,q}(\rho),$$

where ρ is given by (2.32) . ∎

Theorem 2.7. Let q be a convex function with $q(0) = 1$ and h a function such that

$$h(z) = q(z) + zq^{'}(z) \quad (z \in \mathbb{U}).$$

If $f \in \mathcal{A}$, then the following subordination

$$\left(\zeta_q^{\mu} f(z)\right)^{'} \prec h(z) \tag{2.34}$$

implies that

$$\frac{\zeta_q^{\mu} f(z)}{z} \prec q(z), \tag{2.35}$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{\zeta_q^{\mu} f(z)}{z} \tag{2.36}$$

Differentiating (2.36), we have

$$\left(\zeta_q^{\mu} f(z)\right)^{'} = p(z) + zp^{'}(z). \tag{2.37}$$

If we calculate $p(z)$, then we obtain an

$$\begin{aligned} p(z) &= \frac{\zeta_q^{\mu} f(z)}{z} \\ &= \frac{z + \sum\limits_{k=2}^{\infty} \phi_{k-1} a_k z^k}{z} \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in \mathcal{H}[1,1])\,. \end{aligned} \tag{2.38}$$

Using (2.38) in the subordination (2.34) we have

$$p(z) + zp^{'}(z) \prec h(z) = q(z) + zq^{'}(z).$$

Hence by applying Lemma 1.5 , we conclude that

$$p(z) \prec q(z)$$

that is,

$$\frac{\zeta_q^{\mu} f(z)}{z} \prec q(z).$$

and this result is sharp and q is the best dominant. ∎

Example 2.8. If we take $\mu = 0$ in equality (1.9) and $q(z) = \frac{1}{1-z}$ in Theorem 2.7, then

$$h(z) = \frac{1}{(1-z)^2}.$$

and

$$\zeta_q^0 f(z) = z\partial_q f(z) = z + \sum_{k=2}^{\infty}[k,q]a_k z^k \tag{2.39}$$

Differentiating 2.39 with respect to z, we get

$$\begin{aligned}\left(\zeta_q^0 f(z)\right)' &= 1 + \sum_{k=2}^{\infty} k[k,q]a_k z^{k-1} \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in H[1,1])\end{aligned} \tag{2.40}$$

By using Theorem2.7 we have

$$\left(\zeta_q^0 f(z)\right)' \prec h(z) = \frac{1}{(1-z)^2}$$

implies

$$\frac{\zeta_q^0 f(z)}{z} \prec q(z) = \frac{1}{1-z}.$$

Example 2.9. If we take $\mu = 1$ in equality (1.9) and $q(z) = \frac{1}{1-z}$ in Theorem 2.7, then

$$h(z) = \frac{1}{(1-z)^2}.$$

and

$$\zeta_q^1 f(z) = f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2.41}$$

Differentiating 2.39 with respect to z, we get

$$\begin{aligned}\left(\zeta_q^1 f(z)\right)' &= 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in H[1,1])\end{aligned} \tag{2.42}$$

By using Theorem2.7 we have

$$\left(\zeta_q^1 f(z)\right)' \prec h(z) = \frac{1}{(1-z)^2}$$

implies

$$\frac{\zeta_q^1 f(z)}{z} \prec q(z) = \frac{1}{1-z}.$$

Theorem 2.10. Let

$$h(z) = \frac{1 + (2\xi - 1)z}{1+z} \quad ,z \in \mathbb{U}$$

be convex in $\mathbb{U}$, with $h(0) = 1$ and $0 \leq \xi < 1$. If $f \in \mathcal{A}$ satisfies the differential subordination

$$(\zeta_q^\mu f(z))' \prec h(z) \tag{2.43}$$

then

$$\frac{\zeta_q^{\mu} f(z)}{z} \prec q(z) = (2\xi - 1) + \frac{2(1-\xi)\ln(1+z)}{z}. \tag{2.44}$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\zeta_q^{\mu} f(z)}{z} \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in H[1,1]) \end{aligned} \tag{2.45}$$

Differentiating (2.45), we have

$$(\zeta_q^{\mu} f(z))' = p(z) + zp'(z). \tag{2.46}$$

Using (2.46) , the differential subordination (2.43) becomes

$$(\zeta_q^{\mu} f(z))' \prec h(z) = \frac{1 + (2\xi - 1)z}{1+z}.$$

By using lemma(1.3) ,we deduce

$$\begin{aligned} p(z) &\prec q(z) = \frac{1}{z}\int h(t)dt \\ &= (2\xi - 1) + \frac{2(1-\xi)\ln(1+z)}{z} \end{aligned}$$

Using the relation (2.45) we obtain desired result. ∎

Corollary 2.11. If $f \in \Re_{\mu,q}(\xi)$ then

$$\Re\left(\frac{\zeta_q^{\mu} f(z)}{z}\right) > (2\xi - 1) + 2(1-\xi)\ln(2).$$

Proof. If $f \in \Re_{\mu,q}(\xi)$ then from definition(1.2)

$$\Re\left\{\left(\zeta_q^{\mu} f(z)\right)'\right\} > \xi, \quad (z \in \mathbb{U})$$

which is equivalent to

$$(\zeta_q^{\mu} f(z))' \prec h(z) = \frac{1 + (2\xi - 1)z}{1+z}$$

Using Theorem(2.10) , we have

$$\frac{\zeta_q^{\mu} f(z)}{z} \prec q(z) = (2\xi - 1) + \frac{2(1-\xi)\ln(1+z)}{z}.$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we deduce that

$$\Re\left(\frac{\zeta_q^{\mu} f(z)}{z}\right) > \Re q(1) = (2\xi - 1) + 2(1-\xi)\ln(2).$$

∎

Theorem 2.12. Let q be a convex function such that $q(0) = 1$ and let h be the function

$$h(z) = q(z) + zq^{'}(z). \quad (z \in \mathbb{U})$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$\left(\frac{z\zeta_q^\mu f(z)}{\zeta_q^\mu \mathcal{G}(z)}\right)^{'} \prec h(z), \quad (z \in \mathbb{U}) \tag{2.47}$$

then

$$\frac{\zeta_q^\mu f(z)}{\zeta_q^\mu \mathcal{G}(z)} \prec q(z), \qquad (z \in \mathbb{U})$$

and this result is sharp.

Proof. For the functionb $f \in \mathcal{A}$, given by the equation (1.1), we have

$$\zeta_q^\mu \mathcal{G}(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} \frac{\gamma+1}{k+\gamma} a_k b_k z^k, \quad (z \in \mathbb{U})$$

Let us consider

$$\begin{aligned} p(z) &= \frac{\zeta_q^\mu f(z)}{\zeta_q^\mu \mathcal{G}(z)} = \frac{z + \sum\limits_{k=2}^{\infty} \phi_{k-1} a_k b_k z^k}{z + \sum\limits_{k=2}^{\infty} \phi_{k-1} \frac{\gamma+1}{k+\gamma} a_k b_k z^k} \\ &= \frac{1 + \sum\limits_{k=2}^{\infty} \phi_{k-1} a_k b_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty} \phi_{k-1} \frac{\gamma+1}{k+\gamma} a_k b_k z^{k-1}} \end{aligned}$$

We get

$$(p(z))^{'} = \frac{(\zeta_q^\mu f(z))^{'}}{\zeta_q^\mu \mathcal{G}(z)} - p(z) \frac{(\zeta_q^\mu \mathcal{G}(z))^{'}}{\zeta_q^\mu \mathcal{G}(z)} \tag{2.48}$$

Then

$$p(z) + zp^{'}(z) = \left(\frac{z\zeta_q^\mu f(z)}{\zeta_q^\mu \mathcal{G}(z)}\right)^{'}, \quad (z \in \mathbb{U}) \tag{2.49}$$

Using the relation (2.49) in the inequality (2.47), we obtain

$$p(z) + zp^{'}(z) \prec h(z) = q(z) + zq^{'}(z)$$

and by using Lemma (1.5) we get

$$p(z) \prec q(z),$$

that is,

$$\frac{\zeta_q^\mu f(z)}{\zeta_q^\mu \mathcal{G}(z)} \prec q(z).$$

∎

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